

# Students opening up dimensions of variation for learning negative numbers

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## Abstract

*This paper will focus on how the content was handled in the interaction between students and a teacher in 7<sup>th</sup> grade (13 year old students) during a single lesson about addition and subtraction of negative numbers. The study is a development and a follow-up study of learning studies conducted by a research group at Gothenburg University and could be described as a teaching experiment. The aim of this analysis is to see if and how the students contribute to bringing out critical features for learning negative numbers. The result shows that a particular student's questions make an impact on the critical features brought up during the lesson and hence what it is possible to learn. In a previous study four critical features for students' learning about negative numbers were identified. In this new study the teacher was deliberately not bringing up all critical features of the object of learning, to understand negative numbers, in the lesson. The tentative analysis points to the fact that this created a space for students' questions about the object of learning. In this lesson it is therefore the students that are opening up some of the dimensions of variation for learning negative numbers.*

**Keywords:** teaching and learning, learning study, negative numbers, variation theory, design-based research, lesson design

## Introduction

This paper presents an analysis from an ongoing project in which different lesson designs are replicated. In this study the same teacher conducts two lessons in different classes with similar, but in some aspects different lesson designs. The lesson designs are signified by their content specific *critical features*. The *critical features* are seen as essential for learning the topic, in this case addition and subtraction of negative numbers. In this study one lesson with two of these critical features and another lesson with all four critical features was implemented. The expected outcome of these lessons was that the effect on the students learning would differ, since in one of the lessons two of the assumed critical features were missing. Surprisingly the results show that in this particular study the students' learning outcomes in both classes was similar. In the analysis of the

lessons it is shown that the two lessons were actually both bringing out the same critical features in different ways.

### **Method and design of the study**

The aim of this study is to investigate the different lesson designs in different classes. The study could be described as a teaching experiment (Cobb, 2000), a design study (Cobb *et al.*, 2001), or as design-based research (Cobb *et al.*, 2003; Collins, 1992). This study was conducted with one teacher and two groups of students in 7<sup>th</sup> grade (13 year old students). One of these two groups (class A) was more familiar to the teacher than the other group (class B). Both classes were on the same school and could be described as equal in school performance.

The different lesson designs originated from a previous study (Kullberg, 2004) and are described more in detail further on in this paper. In what way the replication of the lesson designs should be done and the detail planning of the lessons was carefully planned in collaboration with the teacher before the implemented lessons. The teacher met the researcher for two hour meetings before each implemented lesson to plan the lesson. In class A, Lesson Design 1, with two critical features and in class B, Lesson Design 2, with four critical features was implemented. Both lessons were about 60 minutes each. The lessons and the meetings were video recorded and large parts of the material have been transcribed. In the excerpts from the transcriptions the symbol (...) is used to show a three second pause while the symbol / is used for interrupted talk and // for overlapping talk. A pre- and post-test was used to measure the learning outcomes. The test consisted of several tasks covering both addition and subtraction with positive and negative numbers, for example to “subtract two negative numbers” ( $-2 - -3 =$  ) or to “subtract one positive and one negative number” ( $2 - -3 =$  ). The pre-test was conducted a week before the implemented lesson and the post-test two days after the lesson. The analysis of the tests was done both on the whole test and on item -level.

The analysis of the lessons has been made by using Variation theory. This theory offers analytical tools that can explain what it is possible to learn during the lessons. Runesson (2005) expresses that “to learn something certain conditions must be met. Although you cannot guarantee that learning will occur if these are fulfilled, at least you can say whether or not necessary conditions for learning were present in the learning situation, and thus whether or not a particular learning was possible” (Runesson, 2005 p 69).

### **Theoretical framework**

In this study a theoretical framework called Variation theory is used. Variation theory (Marton *et al.*, 2004) has developed from phenomenographic research (Marton & Booth, 1997) and could be described as a theory about how we learn and experience the world around us. Within this framework learning is seen as differentiation, in contrast to other theories of learning, for example where learning is seen as adding or constructing new knowledge to the previous (constructivism). In the Variation theory framework learning is to discern specific features of an *object of learning*. People discern different features and therefore have different learning outcomes. Variation is a main concern in this

theory, since to be able to discern an aspect it must be varied in order to be noticed. Every concept, situation or phenomena have particular features or aspects and if an aspect is changed or varies and another remains unchanged, the changed aspect will be noticed. This makes the theory useful also for teachers since teachers could make it possible for the students to experience necessary variation and invariance for a particular object of learning, and its *critical features*. Critical features could be found in the analysis of pre- and post-tests and/or in the analysis of lessons. Research literature, that explores students' understanding of different concepts, could also contribute to the finding of critical features. Though, it is not beneficial to learning to just tell the students the critical features, these must be discerned by the learner. From this follows that the teacher must direct the students' attention to the critical features and if the student can not differentiate between specific features, he or she will have difficulties learning. Therefore experiencing variation concerning critical features is, according to Variation theory, essential for learning.

What the students are able to discern is the variation that is created or present in a learning situation - different *dimensions of variation*. Marton et al (2004) describes this with the following example; If you have a cup on a table you experience different *dimensions of variation* (d.o.v) of the cup. A d.o.v. is in this case seen as different things about the cup that can vary. The color, the material and the shape are all examples of d.o.v., which could vary since the cup could be made of another material, color and shape. In a mathematical context these dimensions of what could vary are connected to the content specific critical feature and how a variation for discerning them could be created.

### **Negative numbers**

Many teachers and researchers agree up on that students often have a hard time to learn addition and subtraction with negative numbers (Ball, 1993). Teachers often think it is a hard topic to teach without falling back on meaningless rules and tricks. In a previous study about negative numbers conducted in Sweden the teachers together with researchers found four critical features that were important for the students understanding in 7<sup>th</sup> and 8<sup>th</sup> grade (Kullberg, 2006; Maunula, in progress). These critical features should be seen as critical for the group that was investigated. Although, it is reasonable that they would be critical even for other learners within the same age and experience. The first critical feature concerns the *difference between the operational sign for subtraction and the sign for a negative number*. In Sweden this sign looks the same, and could easily be misinterpreted. Therefore it is of importance to discern this difference between the signs (Ball, 1993; Gallardo, 1995; Vlassis, 2004). The second feature that was found concerns the way to see subtraction. This implies that when counting with negative numbers it is preferred *to see a subtraction as 'a difference' or 'comparison' between to numbers, instead of (the most common metaphor), 'take away'*. The third aspect also concerns subtraction, since in subtraction there is an in-built perspective, which we seldom speak about. With *'the perspective in subtraction'*, I mean that when counting for example  $(-2)-(-3)=$ , it is the difference between  $(-2)$  and  $(-3)$  seen from the perspective of the  $(-2)$ . Since  $(-2)$  is the bigger number,  $(-2)-(-3)$  equals 1. The forth aspect has to do

with the *understanding of the number system*. In the previous example, it is necessary to know which number is the bigger number, (-2) or (-3)? Some students' don't share this view since they think (-3) is the bigger number. Of course (-3) is a bigger negative number, a larger debt, but (-2) is a bigger number in the number system. In this study two different lesson designs was made from the analysis and the critical features identified from the previous study (Kullberg, 2006; Maunula, in progress).

### **Analysis of the lessons**

The analysis of the lessons show that both lessons, enacted in the classroom were similar in terms of the critical features brought up in the lessons. In the following analysis I will show why this was the case and also indicate how this contributed to the students' opportunity to learn. I have chosen to give a more detailed description of one of the lessons, Lesson 1 (L1). The reason for this is because Lesson two (L2) more closely followed the lesson design that was intended than L1 did. In this part I will show how the content was handled in L1 in the interaction between the students and the teacher. I will show how a particular student's questions opened up for a possibility of discerning critical features for learning about negative numbers.

In L1 it was planned that only two of the four identified critical features would be brought up during the lesson. These features were to see '*subtraction as a difference*' and '*the perspective in subtraction*'. In L2 on the other hand all four aspect were intended to be brought up in lesson and these features were also enacted in the lesson. Although the lessons were intended to be different concerning the critical features they became similar. Important differences between the two lessons were the ways in which the critical features were brought up.

The introduction of both lessons started with the teacher talking about the number line and introducing both positive and negative numbers. This was done by asking the students what was on the other side of zero on a number line. After this the concept of 'opposite numbers' was introduced (in both lessons). In this part the teacher asked for the opposite number, (the number on the other side), for example to the number (plus) five. The students worked some time with finding numbers that were examples of pairs of opposite numbers (e.g. 5, -5). The purpose of this was to later show that when adding the two numbers the sum is zero ( $5 + -5 = 0$ ).

In both lessons the teacher used examples of two persons having a 'shared economy', to introduce addition of both positive and negative numbers. For example, if Lisa has 6 Swedish crowns (sw.cr.) and Bo has a debt of 4 sw.cr., if we look at their 'shared economy' they will have 2 sw.cr. together ( $6 + -4 = 2$ ). This metaphor of negative numbers as debts was also used to introduce subtraction with negative numbers. In connection with subtraction the critical feature of '*seeing subtraction as a difference*' was introduced by the teacher. Though, when using subtraction it was not seen as a 'shared economy', but instead that they compared their economies. If, for example Lisa has 5sw.cr. and Bo -10sw.cr. ( $5 - -10 = 15$ ), Lisa has 15 sw.cr. more than Bo, in this case. The feature of '*the perspective in subtraction*' was in this part discussed since there is diversity between, for example,  $5 - -10 = 15$  and  $-10 - 5 = -15$ .

In L2 two other features of negative numbers, the '*understanding of the number system*' and the '*difference between the signs*' for the operation subtraction and the sign for a negative number was brought up during the lesson. In L1 on the other hand features of negative numbers that was not intended by the teacher was brought up. These features were similar to the critical features brought up in L2. These features brought up made it possible for the students to discern other aspects of negative numbers.

### **The number system: Differences between the positive and negative numbers (L1)**

In L1 the teacher opened up a feature of negative numbers that was not present in the same way in L2. As is shown in excerpt 1 (line 1) below the teacher asks, *is there a difference between these two numbers, we settle with these (-500 and 500), or is it the same number?* By raising this question the teacher opens up for a discussion about the difference between positive and negative numbers (see excerpt 1). This discussion also opens for bringing up what happens to a positive and a negative number when you add the same number to them. In this excerpt the student Felicia reasons with the teachers and says that -500 and 500 are not the same number because if they were the same number and you add seven to both numbers the answer would be the same (see line 4 to 8). So, by contrasting the numbers -500 and 500 it was possible for the students to discern differences between the positive and negative numbers, namely that when you add something to a positive or a negative number it becomes a bigger number. This is shown by that the positive number ( $500 + 7 = 507$ ) becomes more positive while the negative number ( $-500 + 7 = -493$ ) is less negative (see line 11-21).

Excerpt 1.

1. Teacher: Is there a difference between these two numbers, we settle with these [-500 and 500] (...) or is it the same number...what do you say. Any thought about that. Hmm [Swedish: Är det någon skillnad på dom här två talen om vi nöjer oss med dom här...[-500 och 500] (ohörbart) eller är det samma tal (...) vad säger ni. Någon tanke om det. Hm
2. Boy: The difference is thousand. [Swedish: Det skiljer tusen.]
3. Teacher: The difference is a thousand between them. The difference between these numbers five hundred this way and five hundred that way. Is it the same number? (...) Felicia [Swedish: Det skiljer tusen dom emellan. Skillnaden mellan dom här talen är femhundra dit och fem hundra dit. Är det samma tal?(...) Felicia]
4. Felicia: (inaudible) (ohörbart)
5. Teacher: Not (...) why? [Swedish: Inte (...) varför?]
6. Felicia: Because if you add seven for example on both. [Swedish: För att om man för att om man skulle plussa sju till exempel på båda]
7. Teacher: Ahaa
8. Felicia: If it were the same number it would be the same. [Swedish: Om det var samma tal så skulle det bli samma]
9. Teacher: So it wouldn't be the same. [Swedish: Så skulle det inte bli samma]
10. Felicia: If it were the same number it would be the same. [Swedish: Om det är samma tal skulle det bli samma]
11. Teacher: Aha (...) are you following her? She said that if I add seven on that side and then you think you have a number line that is so long, then we have the zero there and we have minus five hundred there and plus five hundred over

there (...) and then I add seven on that side (...) [writes  $-500+7$ ] what do I get? [Swedish: Aha (...) är ni med på vad hon sa? Hon sa om jag plussar sju på den sidan där och då tänker du att du har en tallinje som är sådär jättelång, så har vi nollan där och så har vi minus femhundra där och så har vi plus femhundra här borta så (...) och så plussar jag på sju på den sidan där (...) [skriver  $-500+7$ ] vad får jag då för något?]

12. Felicia: Do I get more minus then? [Swedish: Får jag mer minus då?]

13. Teacher: What did you say? [Swedish: Vad sa du?]

14. Felicia: Do you get more minus then? [Swedish: Får du mer minus då?]

15. Teacher: Yes what do I get then?(...) what do you think? [Swedish: Ja vad får jag då?(...) vad tycker ni?]

16. Boy: On the positive side you get five hundred and seven (inaudible) [Swedish: På plussidan får du femhundrasju. (ohörbart)]

17. Teacher: Then I jump to five hundred and seven on that side (...) and here you say (...) that I have, I will come to get (...) four hundred and ninety three. Minus in front of? [Swedish: Då hoppar jag till femhundrasju på den sidan (...) och här säger du (...)att jag har, jag kommer till att få (...)fyrahundranittitre. Minus framför eller?]

18. Boy: Ah

19. Teacher: How do you think when you think like that? [Swedish: Hur tänker du när du tänker så?]

20. Boy: (inaudible) lies on the minus side (...) so one comes seven plus (inaudible). [Swedish: (ohörbart) ligger på minussidan (...)så kommer man sju plus (ohörbart)]

21. Teacher: Mmm. So you go closer that way and go seven steps so (..) so they they are different numbers here then. [*The teacher show, on the number line, that adding seven means that one goes seven steps to the right on both the positive and the negative number.*] [Swedish: Mmm Så du går närmare ditåt och går sju steg så (..) så det det är olika tal det här då.] [Läraren visar, på tallinjen, att när man adderar sju så går man sju steg åt höger både på det positiva och det negativa talet.]

In this example the variation and invariants that is present in the situation is that the sign (-) before the number 500 is varied with a positive and a negative 500, while the added 7 is invariant. The answers to the tasks are different, they vary. Since it is possible to discern what varies it is possible to discern that -500 and +500 are not the same number.

$$-500 + 7 = -493$$

$$+500 + 7 = +507$$

In conclusion, in this excerpt it is the teacher and the student Felicia that opens up for a possibility for discerning a critical feature, the understanding of the *number system*, namely that the numbers becomes bigger the more on the right side of the number line they are.

### **Differences between the signs and between addition and subtraction (L1)**

In L1 one of the students, Felicia, raises a question about the way of writing the tasks for addition and subtraction. She argues that it is not necessary to use all the signs, for example argues Felicia that  $-8 + 10$  could be written as  $10-8$  instead (line 22-32). This talk is during a group activity and only the students in that group were able to hear it. Later and a little bit provocative she raises the

question to the teacher (in the whole class discussion) about the importance of the order of the written symbols (line 47). *Or doesn't it matter in what order all the signs are?* By the question raised the teacher must go deeper into the meaning of the written symbol language.

Excerpt 2. (Line 22 to 32 in small group discussion, line 33 to 52 in whole class discussion)

22. Felicia: But Jim (the teacher) the thing ...really you could only write nine minus five [Swedish: Men Jan grejen.. egentligen kan man ju bara skriva nio minus fem]

23. Teacher: Ah (...) you could do that [Swedish: Ah (...) det kan man göra]

24. Felicia: And here you could really only write ten minus eight. [Swedish: Och här kan man egentligen bara skriva tio minus åtta.]

25. Teacher: If you understand what happens, you could do that. [Swedish: Om du förstår vad som det är händer, så kan du göra det.]

26. Felicia: Why should we have all these crosses and lines and [Swedish: Varför ha alla dom här korsen och strecken å]

27. Teacher: It is because to be able to tell (...) what what does the plus sign tell us about there really. [Swedish: Det är bara för att tala om att (...) vad vad talar plustecknet om där egentligen.]

28. Felicia: That you should add it. [Swedish: Att man skall lägga ihop det.]

29. Teacher: Ah/

30. Felicia: / But, why should you add it when it becomes the same answer if you take minus only two numbers and a line [Swedish: /Men, varför lägga ihop det när det blir samma svar om man tar minus bara två tal och ett streck]

31. Teacher: We can test it later when we take subtraction later then you could see. [Swedish: Vi kan ju testa sen när vi ska ta subtraktionen sen så kan du se]

32. Felicia: But it is the same thing if you take ten minus eight it becomes two. [Swedish: Men det är ju samma sak om man tar tio minus åtta så blir det två.]

*Twentyfive minutes later*

33. Teacher: If I turn the numbers then. Bo is lying seven crowns behind, what is the difference between there how much money they have now then? (...) what is the difference between these  $(-7 - +4=)$  (...) we are thinking the number line again then. Minus seven, a zero and here plus four. [Swedish: Om jag vänder på siffrorna då. Bengt ligger sju kronor back, vad är skillnaden mellan deras hur mycket pengar dom har nu då?(...) vad är skillnaden mellan dom här  $(-7 - +4=)$  (...) vi tänker tallinjen igen då. Minus sju, en nolla och här är plus fyra.]

34. Boy: Minus eleven. [Swedish: Minus elva.]

35. Teacher: So you mean that the eleven is still there. [Swedish: Så menar på att elvan är fortfarande kvar.]

36. Felicia : // No [Swedish: //Nä]

37. Boy: // if one (inaudable) as he says. [Swedish: // om man (ohörbart) som han säger.]

38. Teacher: Ah you put a minus in front (...) why is there a minus in front? [Swedish: Ah så sätter du minus framför (...) varför är det minus framför?]

39. Boy: Because (...) he is so much behind. [Swedish: För (...) att han ligger så mycket efter.]

40. Teacher: He is in a worse position (...) than she is (...) so it's minus eleven (...) from his point of view. [Swedish: Han ligger sämre(...) till än vad hon gör (...) så det blir minus elva (...) ur hans sätt att se det.]

41. Felicia: Then it must be (...) plus minus four (...) instead. [Swedish: Då måste det vara (...) plus minus fyra (...)]

istället.]

42. Teacher: Excuse me (The teacher doesn't hear). [Swedish: Ursäkta (Läraren hör inte vad eleven säger)]

43. Felicia: In that case it should be plus minus four (...) [Swedish: I så fall skall det vara plus minus fyra (...).]

44. Teacher: (...) Okey (The teacher writes  $-7 + -4 =$  on the board) [Swedish: (...) Okej (Läraren skriver  $-7 + -4 =$  på tavlan)]

45. Felicia: (...) isn't it? [Swedish: (...) eller hur?]

46. Teacher: Ah (...) but that could be (...) certainly. [Swedish: Ah (...) men det kan väl va (...) visst.]

47. Felicia: Or doesn't it matter in what order all the signs are? [Swedish: Eller spelar det ingen roll vilken ordning alla teckena sitter i?]

48. Teacher: But here you say or this is how you say they are only two different examples actually they will give the same answer. It is a very good example that you bring up. If I say but now you are talking about how much they have together when they lived together. He is seven crowns behind (in debt). [Swedish: Fast här säger du eller så här säger du det är bara två olika exempel egentligen dom kommer att ge samma svar. Det är ett jättebra exempel som du tar upp. Om jag säger fast nu pratar du om hur mycket dom hade tillsammans när de bodde tillsammans. Han ligger sju kronor back.]

49. Felicia: But if we wonder how much they have together we can't use minus. If we wonder how much they have together we couldn't use the first example. [Swedish: Om vi undrar hur mycket dom hade tillsammans kan vi ju inte använda minus eller. Om vi undrar hur mycket dom hade kan vi ju inte använda det första exemplet.]

50. Teacher: No, but// [Swedish: Nej utan //]

51. Felicia: //Exactly [Swedish: //Precis]

52. Teacher: The first example is about how big the difference is between their economies when they live apart from each other and that is eleven crowns difference and she lives over there and she has plus four crowns and (...) he is seven crowns behind. From his point of view the difference is minus eleven. But in this case then/ [Swedish: Utan det första exemplet handlar om hur stor är skillnaden i deras ekonomier när de bor ifrån varandra och den är elva kronor skiljer det och hon bor där borta hon har plus fyra kronor han (...) han ligger sju kronor back. Från hans sätt att se det så är skillnaden då däremellan minus elva . Men i det här fallet här då/]

The excerpt shows that in both addition and subtraction the written symbols cause confusion. The teacher tries to make this clearer by contrasting the written symbol language for addition and subtraction by connecting them with two situations, the 'shared economy' with addition and the 'comparison between their economies' with subtraction. Felicia, on the other hand, is by raising questions about the use of symbols pointing to the fact that different tasks (way of writing symbols) give the same answer (the answer is invariant and the meanings of the symbols varies). By doing this she makes it possible to discuss differences between addition and subtraction that gives the same answer. She is by raising questions about, for example that  $-8 + 10$  gives the same answer as  $10 - 8$ , opening up for a possibility of *discerning aspects of the different signs* used in the operations. The discussion raised questions concerning the meaning of the two operational signs + and – and whether it matter what signs are used in an operation.

**Subtraction as a difference:** The difference between 'absolute difference' and the answer (L1)

In both lesson (L1 and L2) the teacher introduced a new way to see subtraction. This way is to see subtraction as a ‘difference’ between two numbers instead of the most common metaphor, ‘take away’. The teacher introduced the concept (difference) by having the students looking at the differences between two numbers on a number line (the absolute difference). By doing this task with the number line all differences becomes positive or ‘absolute’, for example, the difference between a positive four and a minus four is eight. This way of looking at the ‘absolute difference/value’ stays with at least one student when the teacher wants the students to also take ‘the perspective for subtraction’ in consideration when calculating a task. In the excerpt below (line 53-69), a student, Felicia, shows confusion over this use of (absolute) difference and the difference as an answer to a task. She expresses that to the teacher *But Jim, the answer is not the difference the answer is the answer minus or plus* (line 53) She holds on to her statement and says that *The answer is the answer. The answer is minus three the difference is three. It is not the same thing* (line 69).

Excerpt 3

53. Felicia: /But Jim, the difference is not the answer the answer is the answer minus or plus (...) [Swedish: /Men Jan, svaret blir ju inte skillnaden svaret blir ju svaret alltså minus eller plus (...)]

54. Teacher: Ah

55. Felicia: But the thing on that one up there seven and four (..) it is not the difference that is minus three. [Swedish: Alltså grejen på den där uppe sju och fyra (..) så är det ju inte minus tre som är skillnaden.]

56. Teacher: No the difference is three. [Swedish: Nä skillnaden är tre.]

57. Felicia: Ah

58. Teacher: But from his point of view. It (..) he is three crowns behind her compared to Lisa there. [Swedish: Men ur hans sätt att se på det. Det (..) han ligger tre kronor sämre till än Lisa där.]

59. Felicia: You said that the answer should be the difference. [Swedish: Du sa ju att svaret skulle bli skillnaden.]

60. Teacher: What did you say? [Swedish: Vad sa du?]

61. Felicia: You said that the answer should be the difference. [Swedish: Du sa ju att svaret skulle bli skillnaden.]

62. Teacher: Ah the difference is three between these. But one must see from which point of view from his point of view from Bo’s point of view/ [Swedish: Ah det skillnaden är tre här emellan. Men man måste titta ur vem sätt att se på det ur hans sätt ur Bengts fall/]

63. Felicia: / But it is always from the first [Swedish: /Men det är ju alltid den första man]

64. Teacher: Ah it is always from the first you look at [Swedish: Ah det är alltid den första man får titta på]

65. Felicia: Then you can’t say that the answer is difference. [Swedish: Då kan man ju inte säga att svaret blir skillnad.]

66. Teacher: (...) Ohh [Swedish: (...) Öhh]

67. Felicia: It is (becomes) the answer .. [Swedish: Det blir ju svaret ..]

68. Teacher: Ah it is true (...) [Swedish: Ah det är sant (...)]

69. Felicia: The answer is the answer. The answer is minus three the difference is three. It is not the same thing. [Swedish: Svaret blir ju svaret. Svaret är minus tre skillnaden är tre. Det är ju inte samma sak.]

The variation that is brought up in the situation is a contrast between the use of the concept of difference, seen as an absolute value and the answer of a subtraction task. The *difference between absolute value and the answer* is an identified difficulty for the students understanding as is shown in this excerpt. In conclusion, the analysis of L1 show that issues about the understanding of negative numbers, which was not planned by the teacher, were brought up in the lesson. A particular student, Felicia, raised many questions concerning the object of learning. The critical features brought up in L2 t were he intended ones. While in L1, on the other hand, the teacher together with the students opened up dimensions of variation that was not intended on fore hand in the lesson plan.

Table 1. Critical features identified in L1 and L2.

<i>Lesson 1</i>	<i>Lesson 2</i>
<b>The number system</b> (Is 500 and -500 the same number? Adding the same number to a positive and a negative number) <i>excerpt 1</i>	<b>The number system</b> (Is -3 a bigger number than -5?, comparing numbers)
<b>Differences between the signs</b> and between addition and subtraction <i>excerpt 2</i>	<b>Difference between the signs</b> (Is the sign in 5-3 the same as -3?)
<b>The perspective in subtraction</b> (Is 5-3 the same as 3-5?) <i>excerpt 3</i>	<b>The perspective in subtraction</b> (Is 5-3 the same as 3-5?)
<b>Subtraction seen as a difference</b> between two numbers: (The difference between ‘absolute difference’ and the answer) <i>excerpt 3</i>	<b>Subtraction seen as a difference</b> between two numbers

### Analysis of tests

The analysis of the pre-and post-tests show that both classes were equal in performance, since 28% of the students in class A and 30% of the students in class B answered correctly to the items on addition and subtraction before lessons. The post-test shows that both classes improved their results after the implemented lesson. After the lesson 58% of the students in class A and 65% in class B are able to solve the tasks. The classes learning outcomes could be seen as similar. There are still many students, in class A 41% and in class B 35%, that still are not able to solve tasks on the test.

Table 2. Overall results on pre-and post-test.

<i>n</i>	<i>Lesson 1</i>		<i>Lesson 2</i>	
	<i>n=11</i>		<i>n=12</i>	
Task	Pre-test	Post-test	Pre-test	Post-test
Overall result	28%	58%	30%	65%
<b>Change</b>		+30%		+35%

### Conclusions

This study indicates that the two lessons (L1 and L2) became similar, in terms of the critical features brought up. In this study a particular student, in the study named Felicia was struggling to understand the meaning of the signs when dealing with addition and subtraction of negative numbers. The questions raised by Felicia in her interacting with the teacher, striving for understanding, contributes to that the teacher must bring up aspects of negative numbers that was taken for granted by the teacher or that the teacher had not intended to bring up during the lesson. This study could be examples of at least two different phenomena. The first one is as an analysis of how students are opening up dimensions of variation by asking questions about the object of learning. But it could also be seen as an example of the difficulties that a student gets in when the necessary critical features are missing in the lesson. In my analysis of the data I have combined them both. My analysis implies that it is because of the difficulties of understanding that Felicia gets in that makes a space for asking the questions about the object of learning. These questions have an impact on the critical features brought up in the lesson and the dimensions of variation that the students could discern during that lesson. This analysis does not imply that it is in any way better to exclude critical features for learning a particular content in a lesson. But in this special case the questions raised by Felicia made an impact on what it was possible to learn.

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